"Reason, Ratio, *Reductio*"

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A Senior Essay submitted in partial fulfillment of the requirements for the degree of Bachelor of Arts in the Integral Curriculum of Liberal Arts.

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May 3rd, 2019

For books continue each other, in spite of our habit of judging them separately.
-*A Room of One's Own*, Virginia Woolf

To understand a book, it needs to be read as a continuation of the books that precede it. A close look at Kant reveals that he is heavily influenced by mathematics. So, Kant should be read as a continuation of the ideas presented by mathematicians such as Newton and Euclid. Kant's *Critique of Pure Reason* offers a complex argument that exposes flaws in pure reason. In an attempt to untangle Kant's rhetoric, this essay offers a mathematical lens through which Kant can be viewed. Structurally, Kant constructs his proofs similarly to a mathematician. Not only does he classify proof by contradiction in his analysis versus synthesis and *a priori* versus *a posteriori* paradigm, but he also employs proof by contradiction in the explanation of his transcendental aesthetic, or time and space. Regressive synthesis is a tool used to go back through the inner conditions of time and space. Regressive synthesis relies on Newton's revolutionary understanding of division in the *Principia*. To further his understanding of the motion of heavenly bodies, Newton proves that one can look at rectilinear figures to approximate curvilinear motion. The relationship between rectilinear and curvilinear figures Newton relies on can be traced back to Euclid's *Elements*. Using a circle as his example for a curvilinear figure, Euclid proves inscription and circumscription of certain rectilinear figures is possible. Newton uses this as a springboard to prove the relationship between rectilinear figures and curvilinear figures. Once the relationship between Euclid, Newton, and Kant is established, it is fruitful to revisit Kant's antinomies. Using a mathematical lens to analyze Kant's antinomies and the problem reason creates by employing a transcendental illusion sheds light on the complex argument Kant presents. Viewing Kant's work in isolation misses the mathematical foundation Kant relies on to construct his argument.

Kant first mentions mathematics in the introduction of *Critique of Pure Reason*. He declares that all mathematical judgments are synthetic *a priori*. For a judgment to be *a priori* means that there is no experience involved in the judgment. Since mathematical propositions are valid regardless of their application, mathematical judgments are *a priori* (B14). Kant concedes that while some people may not be willing to accept that all mathematical judgments are *a priori*, it is obvious that pure mathematics is *a priori*. Kant calls all mathematical judgments, both arithmetic and geometric, synthetic because of the way mathematics connects two concepts. All judgments, by Kant's definition, connect subject, *A*, to predicate, *B*. For synthetic judgments, "...*B* lies entirely outside the concept *A*, though to be sure it stands in connection with it" (B10). So, synthetic judgments connect two distinct concepts. Kant's example for this is "All bodies are heavy" (B11). Since bodies are merely extension, one does not need weight to think of body. Thus, this judgment connects two separate concepts; therefore, the judgment is synthetic. Although humans' understanding of figures, such as a square or rectangle, is based in experience, mathematics is abstracted from this experience. Calling mathematical judgments synthetic *a priori* is antithetical to Classical views, so Kant spends time justifying this statement.

A method previous mathematicians and philosophers see as analytic is proof by contradiction.[1] This method assumes the given but negates the conclusion to see what follows. Then, the negated conclusion is followed until a contradiction with the given statement is exposed. Because this method is contained within the proof and draws no lines externally, the proof would seem to be analytic; however, Kant calls it synthetic. The justification is that proofs by contradiction rely on other synthetic propositions: "... a synthetic proposition can of course be

[1] Proof by contradiction is the same thing as *reductio ad absurdum*.

comprehended in accordance with the principle of contradiction, but only insofar as another synthetic proposition is presupposed from which it can be deduced, never in itself" (B14). If any portion of a synthetic proposition relies on a synthetic proposition, then the proposition itself is synthetic. Due to the fact that all synthetic proofs connect two distinct concepts, relying on a synthetic proof draws in something more than what one originally began. Although a proof by contradiction may be completely contained, it is deduced from another synthetic proposition since all mathematical propositions are synthetic.

Euclid provides a concrete example of how proofs by contradiction are deduced from other propositions. The first *reductio* in Euclid is Book I proposition 6, which is deduced from proposition 5. Euclid's definition of isosceles triangle says that there are two equal sides (Bk I Definitions). In proposition 5 he proves that given a triangle with two equal sides, the angles subtending those sides are likewise equal (Bk I Prop 5). Then, proposition 6 proves the converse: that given a triangle with two equal angles, the subtending sides are likewise equal (Bk I Prop 6). Although it is self-evident that if equal sides necessitate equal angles; equal angles necessitate equal sides, Euclid must provide a proof. In the proof of proposition 6, he invokes proposition 5 to show a contradiction results if equal angles do not imply equal sides. Proposition 5 is synthetic since it is a geometrical proof. Once proposition 6 relies on proposition 5, it is also synthetic. This proof by contradiction shows how this method is especially useful when the proposition is self evident or, in other words, it easily follows from a previously proven proposition.

Kant employs proof by contradiction throughout the *Critique*. Beyond the introduction, he relies heavily on the logical structure found in mathematics as well as mathematical terminology. In setting up the *Critique*, Kant, like Euclid, lays out all of the definitions in his metaphysical system. A key part of his metaphysical system is the transcendental aesthetic, i.e. time and space, because it is the basis of all human intuitions. Space is outside and time is inside human beings. Time and space are the only *a priori* intuitions. To prove this, Kant not only uses a proof by contradiction, but also employs infinity, which refers to the unbounded nature of both time and space. So, rhetorically and structurally Kant relies on mathematics.

Kant's proof for why space is an *a priori* intuition uses the same logical structure as a proof by contradiction. Given the hypothesis that space is an *a priori* intuition, Kant contradicts the hypothesis and says space is a concept from experience to show that there is a contradiction, which means space must be an *a priori* intuition: "Space is represented as an infinite **given** magnitude. Now one must, to be sure, think of every concept as a representation that is contained in an infinite set of different possible representations...which thus contains these **under itself**..." (B40). Space is understood as a given whole. If space is a concept, then space must be completely contained within itself and received through experience. So, the unbounded nature of space would be bounded: "...but no concept, as such, can be thought as if it contained an infinite set of representations **within itself**. Nevertheless space is so thought...Therefore the original representation of space is an ***a priori*** **intuition**, not a **concept**" (B40). It is impossible that through experience one comprehends something that is both completely contained and infinite. So, space cannot be a concept gained through experience. Instead, it must be an intuition received *a priori*. Further, to relate one object to another, or even to relate oneself to another object, space is presumed. Receiving space as an empirical concept would result in a contradiction because it would require bounding something unbounded. Again, Kant employs proof by contradiction. Beginning by negating the hypothesis that space is an *a priori* intuition, the proof exposes a contradiction and concludes that space must be an *a priori* intuition, as hypothesized. Once Kant proves space is an *a priori* intuition, he goes on to prove time is an *a priori* intuition.

Like space, Kant thinks time is infinite. Humans' attempts at grasping any finite time are simply a form of placing a bound on the infinite scope of time. So, talking about any definite time merely cuts a determinate portion from the infinite whole: "The infinitude of time signifies nothing more than that every determinate magnitude of time is only possible through limitations of a single time grounding it. The original representation **time** must therefore be given as unlimited" (B48). Since receiving a determinate magnitude always requires limiting the whole, time must be given as an infinite whole. As a result, some periods of time, such as from the beginning of time to now, are indeterminate. Further, all human experiences presuppose time. This further shows that time must be given before experience. Since time is given as an infinite whole without experience, it is an *a priori* intuition.

Time is unique because beyond being infinite it also constitutes a series. The rule Kant lays out for the series is based on distinguishing between past, present, and future: "Time is in itself a series (and the formal condition of all series), and hence in it, in regard to a given present, the *antecedentia* are to be distinguished *a priori* as conditions (the past) from the *consequentia* (the future)" (B438). Mathematically, every series has a rule that determines what elements belong in the set. For time, the series is created with the rule that every second is both preceded and followed by another second going to infinity in both directions. Note that this does not necessitate that there is a first second. Finding the beginning of time would be impossible since the human brain is incapable of infinitely recursing through time. So, the linear series constituted by time is infinite since there is no conceivable beginning or end to time that a human could ever find, and time is based solely in the human. Since time is an *a priori* intuition, it is the first series to which humans have access. As such, it is the condition for all other series.

Although space in itself is not a series, the way humans apprehend[2] space places the manifold parts of space in time. By placing space in time, the manifold parts of space constitute a series. "Yet the synthesis of the manifold parts of space, through which we apprehend it, is nevertheless successive, and thus occurs in time and contains a series" (B439). For Kant, space is a reality that all humans experience. For a human to process their senses, it must go through the categories of time and space before becoming an experience, which is further proof that time and space are *a priori* intuitions; therefore, a human's understanding of the object is tied to its relation to space. Since humans require space for experience, it is given as a whole as opposed to as a series. The successive nature of space through apprehension places it in time, which is the formal requirement of all series. Placing space in a series allows Kant to speak of each part of space as a condition of the next. So, every space has boundaries which are the condition of the following spaces. Thus, there can be both a progress and a regress in the series of space. From a starting point one can go forward through the series of conditions as well as backwards through the series of conditions to find proximate and more remote conditions.

Kant wants to consider space as a mathematical series so that one can go through remote conditions of space, which he refers to as regressive synthesis. Reason, however, requires that one go towards the absolutely unconditioned, or the first condition: "Second, reality in space, i.e.

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[2] Apprehension is the process by which objects experienced through sensation becomes images in people's brains, or empirical intuitions. "First of all I remark that by the **synthesis of apprehension** I understand the composition of the manifold in an empirical intuition, through which perception, i.e., empirical consciousness of it (as appearance), becomes possible" (261).

matter, is likewise something conditioned, whose inner conditions are its parts, and the parts of those parts are the remote conditions, so that there occurs here a regressive synthesis, whose absolute totality reason demands;" (B440). Regressive synthesis is the process of using conditions to work from the most proximate consequence to more remote consequences. For reality of space, i.e. matter, regressive synthesis requires a continual division into the inner parts of matter: "and that cannot occur otherwise than through a complete division, in which the reality of matter disappears either into nothing or else into that which is no longer matter, namely the simple. Consequently here too there is a series of conditions and a progress towards the unconditioned" (B440). Regressive synthesis of space, or matter, is a continual division. Since space is given as a whole, the division of space must be from the whole into parts. When reason attempts to progress towards the unconditioned, the division of the whole becomes so small that it either disappears completely or transforms into something other than matter, which Kant calls "the simple". This poses a problem for reason itself because the absolutely unconditioned is unobtainable. Since the infinite series of space continues to infinity in both directions, receiving the first member of the series is impossible. Regressive synthesis, which is based on mathematics, is a tool Kant uses to expose a flaw in the employment of pure reason.

Regressive synthesis definitively connects Newton and Kant's *Critique*.[3] Dividing into something infinitely so that ultimately what is left is something other than what one began with comes directly from Newton dividing a rectilinear figure so it is ultimately no different than a curvilinear figure. The mathematics Kant relies on that is based in Newton sheds light on Kant's critique. By reading Newton, one gains a better understanding of Kant's perspective on

[3] Beyond this, Kant learned natural philosophy and astronomy, which is based in Newton.

regressive synthesis, which shows the mathematical influence on Kant, specifically when he constructs the transcendental aesthetic.

Newton's first lemma claims quantities or a ratio of quantities which tend toward equality are ultimately equal. He proves this with a proof by contradiction: "*Quantities, and also ratios of quantities, which in any finite time constantly tend to equality, and which before the end of that time approach so close to one another that their difference is less than any given quantity, become ultimately equal*" (Bk I Lemma 1). If not, the two quantities would be ultimately unequal such that there is an ultimate difference, D. So, the two quantities would never be closer than D. This, however, is against the hypothesis that the two quantities always tend towards equality. In this proof by contradiction, Newton refers to ultimate ratios.[4] Looking at the ultimate ratio means looking at what the ratios tend towards as they approach a given value. One way to look at this is if, as the quantities and ratios of quantities get larger and larger, the difference between them becomes negligible, then the two quantities are called ultimately equal. Alternatively, in the next lemma, he employs lemma 1 to show that if, after continual division, two figures tend towards equality they are ultimately equal. So, ultimate ratios focus on where the two quantities approach one another, whether that be at a very large number or at the very beginning of motion.

Newton in the following lemma inscribes and circumscribes parallelograms about a curve. The point of this lemma is to show that if the number of parallelograms in and around the curve are increased infinitely, the difference between the parallelograms and the curve is smaller than any given difference and, as such, the figures are ultimately equal. "*...if then the width of these parallelograms is diminished and their number increased indefinitely, I say that the*

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[4] Newton's usage of ultimate ratios is similar to a limit.

ultimate ratios which the inscribed figure...the circumscribed figure...and the curvilinear figure...have to one another are ratios of equality" (Bk I Lemma 2). Infinitely increasing the number of parallelograms makes each of the individual parallelograms smaller. The inscribed parallelograms become closer to the bottom of the curve while the circumscribed parallelograms get closer to the top of the curve. So, the inscribed figure is directly below and the circumscribed figure is directly above the curve. As the number of parallelograms infinitely increases, there is ultimately no difference between the three figures. Lemma 3 proves that it makes no difference if all of the parallelograms are equal or unequal. Either way, infinitely increasing the number of parallelograms results in the rectilinear figure approaching the curvilinear figure.

Kant's regressive synthesis comes from Newton's understanding of division. While Kant uses this argument to further his transcendental aesthetic, Newton uses this technique to analyze planetary orbits. Both authors are experimenting with different tools in an attempt to understand magnitudes that are too large for human understanding through experience. Kant employs infinite regression to analyze the elements of the transcendental aesthetic: time and space. Newton employs infinite division to analyze the movement of heavenly bodies through an analysis of forces.

The laws of motion hold equally for curvilinear and rectilinear figures, Newton uses rectilinear figures to approximate centripetal forces of orbits in the shape of the conic sections. The first example he gives is a body in continual motion that is acted upon by another force in a direction inclined to the motion of the original body: "*A body acted on by [two] forces acting jointly describes the diagonal of a parallelogram in the same time in which it would describe the sides if the forces were acting separately*" (Law 3 Cor 1). The two forces create a parallelogram of forces and the body travels along the diagonal of the parallelogram. After the lemmas, Newton applies the parallelogram of forces to curvilinear figures since there is ultimately no difference between a rectilinear and curvilinear figure. The lemmas prove that curvilinear figures can be described by an infinite number of rectilinear figures. So, an infinite number of parallelograms with a centripetal force pulling inwards would result in the diagonals of multiple parallelograms creating a smooth curve. This method bridges the gap between rectilinear motion and curvilinear motion as well as theoretical and practical mathematics. A body is observed to travel in an arc. Newton proves that this arc can be resolved into an infinite number of parallelograms that describe the forces acting on the body that results in the body moving along a curvilinear path. Since the parallelogram of forces can be used for curves as well, all of the laws of motion hold for curvilinear figures, which means that the laws of motion for centripetal forces are the same as the laws of motion for any other force.

Newton employs infinite diminution to find the centripetal force of different orbits pertaining to conic sections. The first of these propositions uses an argument from just nascent arcs.

> *If in a nonresisting space a body revolves in any orbit about an immobile center and describes any just-nascent arc in a minimally small time, and if the sagitta of the arc is understood to be drawn so as to bisect the chord and, when produced to pass through the center of forces, the centripetal force in the middle of the arc will be as the sagitta directly and as the time twice...inversely* (Bk I Prop 6).

When speaking of centripetal force, Newton discusses ultimate ratios of bodies. By looking at the beginning of motion, or just-nascent arcs in a minimally small time, Newton can apply the properties of rectilinear figures to curvilinear orbits. This allows Newton to give an equation for centripetal force based on relationships between centripetal force, sagittas, and time. Newton then applies the generic equation for centripetal force to each conic section.

After going through each conic section, Newton concludes by finding the course of an orbit given the centripetal force and initial velocity. So, ultimately Newton can determine the course of any orbit with only a snapshot of its movement. "*Supposing that the centripetal force is inversely proportional to the square of the distance of places from the center and the absolute quantity of this force is known, it is required to find the line which a body describes when going forth from a given place with a given velocity along a given straight line*" (Bk I Prop 17). With heavenly bodies, it is impossible to see one revolution in its entirety from the earth. So, Newton uses infinite diminution to be able to find the orbit of a body given only the beginning of its motion. Newton invented a tool to help humans comprehend phenomena that they are incapable of experiencing, namely the orbit of a heavenly body. Similarly, Kant uses regressive synthesis to go through the past conditions of space, something outside of humans that is given as infinite, yet completely contained *a priori*.

Both Newton and Kant recognize a connection between mathematics and philosophy. While infinite division points towards calculus, which Newton was among the first to investigate, *The Principia* is based in geometry. Part of the justification for using geometry is merely because it was more accessible at the time; however, Newton gives a more precise reason for using geometry in the introduction: "Therefore *geometry* is founded on mechanical practice and is nothing other than that part of *universal mechanics* which reduces the art of measuring to exact propositions and demonstrations" (Preface). Geometry extracts from the physical act of drawing lines and circles to solve problems using the abstract notion of a line or a circle. To this end, it is helpful for natural philosophy: "For the basic problem of philosophy seems to be to discover the forces of nature from the phenomena of motions and then to demonstrate the other phenomena from these forces" (Preface). Geometry abstracts from physical motion to exactly describe forces and motion. Then, these measurements with the help of geometry can be used to demonstrate other motions. Thus, Newton applies mathematical principles to natural philosophy. The extension Newton uses from mechanics to general principles comes from Euclid.

Euclidean geometry only requires a straightedge and a compass; however, the understanding is that the physical representations drawn of a point or a line are not a real point nor a real line. For, a point is that which has no part and a line is a breadthless length (Bk I Def), neither of which humans can experience. The easiest example of this is a circle. Although everyone has an understanding of what a circle is, drawing a perfect circle is impossible. So, mathematicians settle for imperfect diagrams to refer to when demonstrating geometric principles. Euclid builds up geometric foundations by beginning with propositions on circles, straight lines, and triangles. Once he builds up the rectilinear figures, he dedicates Book IV to the relationships between rectilinear and curvilinear figures. Beyond the philosophical understanding Newton gains from Euclid, Newton relies heavily upon the connection between rectilinear and curvilinear figures to explain the motion of heavenly bodies.

Euclid's connection between rectilinear and curvilinear figures begins in Book I. His first proposition connects an equilateral triangle to a circle. "*On a given finite straight line to construct an equilateral triangle*" (Bk I Prop I). Beginning with a line, Euclid takes one end as the center of a circle with a radius equal to the straight line and constructs a circle. Then, he takes the opposite end of the line as the center of a second circle with a radius equal to the straight line and constructs another circle. Thus, the radii in both circles are equal. So, from the two straight lines between each center of the circle and the intersection of the two circles as well as the shared radius, an equilateral triangle is constructed. With this construction, Euclid builds a rectilinear figure using a curvilinear figure. This construction; however, is different from a proof. From this proposition, it is clear that an equilateral triangle can be constructed within a circle, but there is no way to prove that this method works for any given triangle. Euclid begins proving the relationship between rectilinear and curvilinear figures in Book IV.

Book IV begins with definitions to explain circumscription and inscription of rectilinear figures and circles. The book first addresses inscription of a rectilinear figure in a circle. "A rectilinear figure is said to be *inscribed in a circle* when each angle of the inscribed figure lies on the circumference of the circle" (Bk IV Definitions). So, inscription is when the rectilinear figure fits inside of the circle such that each corner of the figure is touching the circumference of the circle, and the figure is completely contained within the circle. Circumscription, on the other hand, is when a figure surrounds a given figure. "A rectilinear figure is said to be *circumscribed about a circle*, when each side of the circumscribed figure touches the circumference of the circle" (83). Instead of the corners of the rectilinear figure touching the circle, for circumscription, each side of the figure is tangent to the circle. This results in the rectilinear figure perfectly fitting around, yet never cutting the circle.

In Book IV, Euclid proves connections between rectilinear and curvilinear figures. The first proof allows the reader to fit a straight line equal to a given straight line in a given circle. "Therefore into the given circle *ABC* there has been fitted *CA* equal to the given straight line *D*" (Bk IV Prop 1). The only supposition for the proof is that the given line is less than the diameter of the circle. Using properties of straight lines and circles, Euclid is able to inscribe a straight line equal to a given straight line within the given circle, which results in a chord of the circle. This chord is the foundation for building rectilinear figures within the circle.

After inscribing a straight line within the circle, Euclid demonstrates all inscription and circumscription possibilities with a triangle. First, Euclid inscribes a triangle into a circle. Similar to the line proof, he starts with a triangle outside of the circle and proves he can fit in an equiangular triangle. “Therefore in the given circle there has been inscribed a triangle equiangular with the given triangle” (Bk IV Prop 2). This proof begins with a line drawn tangent to the given circle. Then, from the tangent, angles are constructed equal to the angles of the given triangle. This gives two lines drawn within the circle that is connected by a third constructing a triangle within the given circle. Euclid then proves the triangle is equiangular with the given triangle by relating the inscribed triangle with the tangent. In the next proposition, he circumscribes a triangle equiangular with a given triangle about a given circle using tangents and supplementary angles. The next two proofs inscribe a circle into a given triangle and circumscribe a circle around a given triangle. So, Book IV proves that a connection between a triangle and circle can always be established. Beginning with a given circle and a given triangle, an equiangular triangle can be constructed inside of and around the given circle. Alternatively, beginning with a given triangle, a circle can be either inscribed within or circumscribed about the rectilinear figure.

Euclid quickly scales up from a triangle to a fifteen-sided figure by incrementally adding more sides to the inscribed rectilinear figure. He goes through the same process as the triangle with the pentagon. Then, he inscribes a hexagon inside a given circle. From there, he jumps to a fifteen-sided figure: "Let *ABCD* be the given circle; thus it is required to inscribe in the circle *ABCD* a fifteen-angled figure which shall be both equilateral and equiangular" (Bk IV Prop 16). This proof requires first inscribing a side of both an equilateral triangle and an equilateral pentagon. Then, Euclid uses proportions to dictate how much of the fifteen-sided polygon belongs in each section of the circle. Using proportions, Euclid finds one side of the equilateral figure. Then, once the length of one side is known, from proposition 1 of Book IV, the given straight line can be replicated so that the entire fifteen-sided equilateral polygon is constructed within the circle.

Although Euclid makes no conclusions about continuing this process for a figure larger than fifteen sides, the method can be continued to inscribe an equilateral thirty-sided polygon and so on. This idea is where Newton picks up Euclid's method. It is obvious that as the inscribed figure increases in number of sides, the difference between the rectilinear figure and the circle diminishes. Although Newton does not use a circle, he increases the number of parallelograms infinitely by decreasing the area of each parallelogram around a curve. So, in increasing the number of parallelograms, the number of intersections between the parallelogram and the curve also increases. Further, Newton proves that this relationship holds true even if the parallelograms are not equilateral. Whether or not the parallelograms are equilateral, Newton shows ultimately there is no difference between the rectilinear and curvilinear figure. This new understanding of division influences Kant, which is seen through regressive synthesis. With an understanding of the way Kant uses mathematics in the transcendental aesthetic, this lens can be used on other portions of the critique.

The mathematical framework Kant employs continues throughout the explanation of the transcendental illusion, the antinomies, and the ultimate overturning of the principle of reason, which claims that when the conditioned is given the absolutely unconditioned is also given. Rereading these sections with a mathematical lens helps untangle Kant's *Critique*. The heart of the critique rests in the antinomies, which are pairs of seemingly contradictory statements for which Kant supplies flawless proofs. The second and third antinomies expand upon the first antinomy. The first pair discusses the beginning of time in the world and whether or not space is bounded, the second discusses composite and simple substances using space, and the third discusses causality using time. So, the antinomies begin with a broad metaphysical question about the transcendental aesthetic and then hone into both time and space in further detail. These antinomies are false binaries created by pure reason because of the transcendental illusion used in the principle of pure reason. This portion of the *Critique* can be efficiently analyzed with the understanding that Kant's argument is based in mathematics.

The antinomies are intriguing because Kant offers flawless proofs of two seemingly contradictory statements. All of the proofs are done by contradiction. So, he negates both hypotheses and shows that each lead to a contradiction respectively. The first pair of antinomies proves that the world both has a beginning in time and space, which means time and space are bounded, and the world does not have a beginning in time and space, which means time and space are unbounded (B454-61). The contradiction that results if one assumes that there is no beginning to time and space hinges upon the series constituted by time and space as well as Kant's definition of infinity. For something to be infinite, the series must never be completed; however, if there was no beginning to time, from now to the beginning of time would be a completed infinity, which is impossible. So, there must be a beginning to time. Similarly, for space to be a whole, the synthesis of the parts of space must be completed. Thus an infinite whole would be completed. As such, space must be bounded. On the other hand, if the world had a beginning in time or space, then there would be a "no time" or a "no space", which is impossible. A "no time" is impossible because nothing could transpire in "no time". Nothing could start the first moment since time is nonexistent. A "no space" would result in the world being in relation to an empty space, or nothing. This is likewise impossible, which shows there is no beginning to space. These contradictory proofs show there both is and is not a beginning of time in the world and that space both is and is not bounded. The next antinomy takes a closer look at space.

The second pair of antinomies discuss whether or not composite substances in the world are based on simple parts. If composite substances are based in simple parts, then everything is either simple or composed of simples. If not, then nothing is simple. The usage of simple comes up earlier when discussing space. The reality of space is matter, and the smallest inner parts of matter either are composed of nothing or the simple. The thesis claims that the world consists of simple parts. When Kant negates the thesis, the contradiction that results is nothing would be given. If, after abstracting the composite part of a composite substance, nothing was given at all, then there is no way a composite substance could be given; however, there are composite substances that are given. Thus, everything must be composed of the simple. On the contrary the antithesis claims there is no simple part. The contradiction that results is that by assuming there is a simple, the simple would have to take up space. Further, space is infinitely divisible, which means each portion of space is composed of space. Space, however, is a composite. Thus, the simple would also be a composite, which is impossible. This proof hinges on the assumption that each composite substance humans encounter solely through experience. Since experience can never give an absolutely simple substance, it is nowhere in the world. To question the existence of simple versus composite substances, Kant uses the infinite divisibility of space, which is proved mathematically. The next antinomy highlights the mathematical relationship with time.

The third pair of antinomies discusses causality (B472-9).[5] The thesis proposes that there exists causality based on the laws of nature; however, there is also a causality through freedom. The antithesis claims there is no freedom and the only causality that exists is based on the laws of nature. Both the thesis and antithesis view causality in terms of a series. So, each cause has an effect and each effect comes from a corresponding cause, according to the law of nature.[6] The proof of the thesis shows that without freedom, causality through the laws of nature would be impossible since there could be no way for the series of causes and effects to begin. So, a causality from freedom is required for a causality from the laws of nature to take place. On the other hand, the antithesis shows that causality according to freedom breaks the causal law. Absolute spontaneity means an effect happens that does not correlate with a cause. This, however, does not align with experience. Thus, similar to the antithesis of the second antinomy, it is impossible. To further respond to the thesis, Kant states mathematically there is no reason for a first moment in time. Similarly, there is no need to find a first cause. Thus, the antinomy further expands on the first pair of antinomies, the pair devoted to questioning whether or not there is a beginning of time.

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[5] The question of causality is also known as the first mover problem.

[6] "But now the law of nature consists just in this, that nothing happens without a cause sufficiently determined *a priori*" (484).

Both the second and third antinomies tackle a part of the first antinomy and expand upon it. Part of questioning whether there is a bound to space is what that bound would like. Would space end in simple parts or is having simple parts impossible in reality? Further, if there were a beginning in time, would that first moment be based in laws of nature or transcendental freedom? Not only is the content within these proofs mathematical, but the structure also mirrors mathematics. These pairs of antinomies are not isolated, but are to be taken as a group of proofs that build upon one another. On one side, there are the theses which represents a consistent set of beliefs and on the other is the antitheses. This, however, is a false binary. Each of the antinomies have flawless seemingly contradictory proofs because they are all based in a transcendental illusion.

Transcendental illusions result in erroneous conclusions that are often overlooked. Illusions occur in judgment and lead to error. An error does not occur in an object itself, which means the error occurs neither in the understanding nor the senses. Instead, it occurs in sensibility.[7] Kant focuses on transcendental illusions, which have nothing to do with experience, because transcendental illusions make it seem as if one is heading towards pure understanding by mistaking an objective necessity for a subjective one. In order to expose the illusion, Kant takes a mathematical route:

> In order to distinguish the proper action of the understanding from the force that meddles in it, it will thus by necessary to regard the erroneous judgment of the understanding as a diagonal between two forces that determine the judgment in two different directions, enclosing an angle, so to speak, and to resolve the composite effect into the simple effects of the understanding and of sensibility (385).

[7] "The capacity (receptivity) to acquire representations through the way in which we are affected by objects is called **sensibility**" (172).

Similar to the parallelogram of forces used by Newton, Kant explains the misdirection of judgment as a pull away from the straight line created by the senses. An object is on a straight line towards understanding until it is acted upon by sensibility, which pulls the object away from the understanding. These two actions are resolved by the object travelling along a diagonal resulting in an illusory judgment. Using mathematics to understand transcendental illusions as a parallelogram of forces yields a diagrammatic representation, which makes his explanation more tangible. In a transcendental illusion this comes about by subjective necessity being taken as an objective necessity, which results in a break of logic. Kant shows that the principle of reason is based in a transcendental illusion, which is why contradictions appear abundantly in the antinomies.

The principle of reason is an assumption implicit in all proofs that employ reason. Given a conditioned statement, the principle of reason allows one to receive the whole sum of conditions. "**If the conditioned is given, then the whole sum of conditions, and hence the absolutely unconditioned, is also given,** through which alone the conditioned was possible" (B436). Something is conditioned if it relies on something else which came before it. The example Kant gives is the alphabet. If one considers the alphabet to be a series where each letter is given as conditioned by the letter that comes before it such that n is given as conditioned in respect to m, b is given as conditioned in respect to a etc; then the sum of conditions would be the alphabet up to the letter that is given, which is called the absolutely unconditioned. The absolutely unconditioned only pertains to the series of conditions up to what is given. Even though n could be found, it is not given in the series of conditions for m to be given, so it is not a part of the whole sum of conditions. The reliance on a series allows Kant to connect the principle of reason to both time and space. Reason requires that since the now is given, the entire time between the now and the beginning of time is also given necessarily. Similarly, since parts of space are given as conditioned, the whole of space must also be given. Both of these conclusions rely on reason and were employed in the antinomies; however, these conclusions are based in a transcendental illusion.

Transcendental illusions are created by equivocations. By rewriting a usage of the principle of reason into a syllogism, Kant shows that the major premise is transcendental while the minor premise is empirical. The major premise: if the conditioned is given, the whole series of conditions are also given. The minor premise: the objects of sense are given. The conclusion is that the whole series of the objects of sense are given. The major premise is transcendental, while the minor premise is empirical. The major premise is transcendental because working from conditioned through the whole series of all conditions occurs only in the understanding separate from the potential of appearances.The minor premise, on the other hand, deals with appearances. Since the objects of sense are received through experience, the observation that they are conditioned is empirical. The shared term between the major and minor premise is the given condition, but the way in which "condition" is used is different in the major and minor premise. Thus, drawing a conclusion rests on an equivocation. Once appearances are involved, the principle of reason no longer gives the entire series of conditions. "...I could not presuppose the **totality** of synthesis and the series represented by it...because there [in the transcendental case] all members of the series are given in themselves (without time-condition), but here they are possible only through the successive regress, which is given only through one's actually completing it" (B529). Once time is involved, the completion of regressive synthesis can no longer be taken as a given. Instead, it can only be seen as a problem. Thus, one can attempt to go back through the series of conditions; however, receiving the first member of the series or the whole series of conditions requires working through the past conditions. It is no longer an assumption one can make that the absolutely unconditioned truly exists. This assumption bears on each of the antinomies.

Beyond the principle of reason, the assumption that the world is one whole is a transcendental illusion. All of the antinomies assume that the world is one whole. The first antinomy asks whether or not the world has a beginning in both time and space The second antinomy questions simple versus composite substances in the world. The third antinomy discusses causality, specifically focusing on the first cause of the world. Asserting the world is a whole is based in empirics. The only reason humans can say the world is a complete whole is through a synthesis of the manifold representations of the world. Using this empirical basis to draw a conclusion about the transcendental aesthetic, space and time, requires an equivocation. So, this error bears on all of the antinomies. The implication of this is the antinomies set up two statements that are not true opposites. If one were to assume that the world is not a whole, then asserting whether or not there is a beginning to time and space in the world is meaningless. Similarly, asking whether or not the incomplete world is based in simple or composites would not make sense. Nor would it make sense to find a first cause for an incomplete world. To be a true opposite, the negation of one of the statements must imply that the other statement is true. Once there is a third option, the statements are no longer true opposites. Instead, they are dialectal opposites. Thus, the antinomies do not create the binary that reason claims.

A look at the mathematical influence in Kant helps untangle the problems created by pure reason. Pure reason sets up contradictions presented in the antinomies. Discussing the beginning of the world in time and space and the results of whether or not one believes in a beginning has far reaching consequences philosophically. Kant chooses the third road by showing the contradictions presented are not truly contradictions. To do so requires extensively building up definitions and theorems. Each term is rigorously defined so that the proofs of the antinomies as well as the exposed transcendental illusion hold. A key piece in Kant's foundational work is the transcendental aesthetic: time and space, the only two *a priori* intuitions. Proving that time and space are *a priori* intuitions are proofs by contradiction. Further, calling time and space infinite series cements Kant's relation to mathematics. Since time and space constitute a series, reason demands that the whole series of conditions are given through regressive synthesis, which is a form of infinite division. Newton uses infinite diminution to show that there is ultimately no difference between curvilinear and rectilinear figures so that he can describe the motion of planetary bodies. The relationship between curvilinear and rectilinear figures traces back to Euclid, the founder of modern geometry. Beyond the technical relation, they all employ the same proof by contradiction and see a connection between mathematics and philosophy. Tracing the connection from Kant back through Newton and then to Euclid sheds light on Kant's perspective and makes reading the argument against the antinomies easier. For, as Woolf notes, no book exists in isolation.

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